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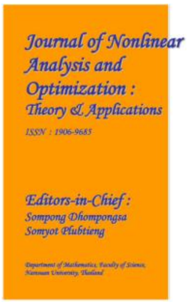

ON CLOSELY CONNECTED DOMINATION IN PRODUCT OF GRAPHS

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Abstract: Let $G = (V(G), E(G))$ be a simple, finite, connected graph. The vertices $x, y \in V(G)$ are closely connected, if at least one of the shortest path connecting them is not a cut path. A set $D \subseteq V(G)$ is called a closely connected dominating set(CC-dominating set), if every vertex $y \in V \backslash D$, there exist some vertex $x \in D$, where x and y are closely connected vertices. The minimum cardinality of a CC- dominating set of G is called the CC-domination number, it is denoted by $\gamma_{cc}(G)$. In this paper we discussed the CC domination number for Cartesian product of paths and cycle graphs with some examples.

***Keywords**:* closely connected vertices, CC-degree of a vertex, domination number, CC-domination number, Cartesian product of graphs.
2020 Mathematics Subject Classification: 05C40,05C07, 05C69, 05C76.

1 Introduction

In this paper we have taken the graphs to be finite, simple and undirected graphs. The domination in graphs is one of the important concept in graph theory. In 1958, the Concept of domination in graphs was defined Claude Berger and ore in [1]. The domination of graphs are discussed in [2] are depending the adjacency property of vertices in a graph. That is, a set $D \subseteq V(G)$ is called dominating set of G, if every vertex $V(G) \backslash D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of its dominating set. Instead of adjacency pair of vertices. K. Priya and V. Anilkumar introduced a new type at vertex pairs are called closely connected vertex pairs in [5].

Let $G = (V(G), E(G))$, the vertices $x, y \in V(G)$ are closely connected, if at least one of the shortest paths connecting them is not a cut path. The Authors K. Priya and V. Anilkumar motivated this concept and introduced the closely connected dominating set in [4]. A set D of $V(G)$ is closely connected dominating set (abbreviated as CC-dominating set), if every vertex $y \in V \backslash D$ is closely connected to some vertex in D. The minimum cardinality of CC-dominating sot is called CC-domination number, and it is denoted by $\gamma_{cc}(G)$. Also $\Gamma_{cc}(x, y)$ is the cardinality of the set of all shortest paths connecting x and y except the cut paths.

The open CC-neighbourhood of a vertex in V is the set $N_{cc}(x) = \{y \in V(G); \Gamma_{cc}(x, y) \geq 1\}$ where as the closed CC-neighbourhood of x is defined as $N_{cc}[x] = N_{cc}(x) \cup \{x\}$.
The cardinality of $N_{cc}(x)$ is the CC-degree of the vertex x, and is denoted by $\deg_{cc}(x)$. The vertex $y \in V(G)$ is said to be CC-Isolated vertex if $N_{cc}(x) = \phi$. The domination in Cartesian product of graphs studied in [3].
In this paper, discuss about the CC-domination number in Cartesian product at paths, cycles and Paths. We begin with some basic definitions. and notations.

Definition 1.1 *A subset D of V is called CC-dominating set if every vertex $y \in V \backslash D$, there exist some closely connected vertex in D.*

Definition 1.2 *The Cartesian product of two simple graphs G and H is denoted by $G \square H$, whose vertex set is $V(G) \times V(H)$ and $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are adjacent if $x_1 = y_1$ in G and x_2 is adjacent to y_2 in H (or) x_2 is adjacent to y_2 in G and $x_2 = y_2$ in H.*

For a graph G of order n, the following are some basic results for $\gamma_{cc}(G)$ in [4].

1. $1 \leq \gamma_{cc}(G) \leq n$.
2. For a Path P of order n. $\gamma_{cc}(P_n) = n$.
3. If G has no cut edge, then $\gamma_{cc}(G) \leq \gamma(G)$.

Theorem 1.3 *[4] Let G be a graph and $u \in V(G)$ be a CC-isolated vertex. Then u belongs to every CC-dominating sets of G.*

Theorem 1.4 *For the cycle $C_n, \gamma_{CC}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$, $\forall\ n \in \mathbb{N}$.*

Proof. In C_n and let $V(C_n) = \{1,2,3,\ldots,n\}$. Every vertex in C_n is closely connected only with its adjacent vertices. So we conclude the above result.

Example 1.5 *Consider the graph C_5 in the Figure 1,the set D={1,3}. is CC- Dominating set.*

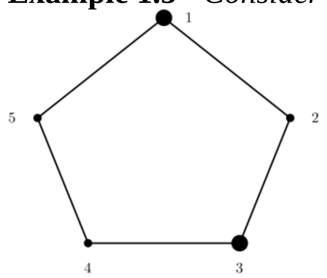

Figure 1: Vertices with bold circles form a minimum CC-dominating set for C_5

2 γ_{cc} in Cartesian Product of Paths

Let P be a the path of order n. The vertices of a path can be listed in the order $\{x_1, x_2, \ldots x_n\}$ such that the edges are $\{(x_1,x_2),(x_2,x_3),\ldots(x_{n-1},x_n)\}$.
Consider the graph P_7. Shown in Figure 2. The vertices set D form a minimum dominating set. i.e., $D = \{x_2, x_5, x_7\}$. Hence $= 3$.

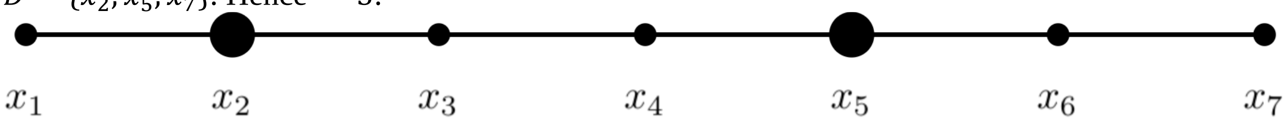

Figure 2: The path with 7 vertices

But every vertex in P_7 shown in Figure 2, are CC -Isolated, Since $N_{cc}(x_i) = \phi$ for $i = 1$, to 7. Every vertex are CC-Isolated. Then the minimum cardinality of CC-dominating set is 7. Hence $\gamma_{cc}(P_7) = 7$.

Theorem 2.1 *[2] For any path, $G = P_n$, then $P(G) = \left\lceil \frac{n}{2} \right\rceil$.*

Theorem 2.2 *[4] For any path, $G = P_n$, then $\gamma_{CC}(G) = n$.*

The following Lemma is useful to prove the further Theorems on CC-dominating sets.

Lemma 2.3 *Let G be a finite graph of order n and for any vertex $x \in V(G)$ with $deg_{cc}(x) = n - 1$, then the $CC-$ domination number $\gamma_{cc}(G) = 1$.*

Proof. Let G be a finite graph of order n. For any vertex $x \in V(G)$ with $\deg_{cc}(x) = n - 1$, which means that $|N_{cc}(x)| = n - 1$. That is vertex x is closely connected with remaining $n - 1$ vertices and conclude that $\Gamma_{cc}(x,y) \geq 1$. $\forall y \in V(G)$ and $x \neq y$. Then $D = \{x\}$ is a CC-dominating set, since every vertex $y \in V \backslash D$ is closely connected, with single vertex x in D. Hence the minimum cardinality of CC-dominating set is 1. i.e, $\gamma_{cc}(G) = |D| = 1$.

Theorem 2.4 *For $n \geq 1$, $\gamma_{cc}(P_2 \square P_n) = \begin{cases} 2, & if n = 1,2 \\ 1, & otherwise. \end{cases}$*

Proof. Let $G = P_2 \square P_n$. The vertices of path P_2 and P_n can be listed in the order $\{1,2\}$ and $\{1,2,3,\ldots,n\}$ respectively. For $n = 1, G = P_2 \square P_1$ is a path with two vertices, By Theorem 2.2 then $\gamma_{cc}(G) = 2$.
For $n = 2$, we have $G = C_4$ is a cycle with four vertices. By using Theorem 1.4, $\gamma_{CC}(G) = 2$. For $n > 2, G = P_2 \square P_n$. In this case $|V(G)| = 2n$. In G, there are four vertices of degree 2 and $2n - 4$ vertices of degree 3. The vertices of degree 2 are $\{(1,1),(1,n),(2,1),(2,n)\}$.
Consider the graph $P_2 \square P_n$ shown in Figure 3. The closely connected neighbourhoods of $(1,1),(2,1),(1,n)$ and $(2,n)$ are

$$N_{cc}(1,1) = \{\{(1,i); i = 2,3 \dots n\} \cup \{(2,i); i = 1,2, \dots n-1\}\}$$
$$N_{cc}(2,1) = \{\{(2,i); i = 2,3, \dots n\} \cup \{(1,i); i = 1,2, \dots n-1\}\}$$
$$N_{cc}(1,n) = \{\{(1,i); i = 1,2, \dots n-1\} \cup \{(2,i); i = 2,3, \dots n\}\}$$
$$N_{cc}(2,n) = \{\{(2,i); i, 2, \dots n-1\} \cup \{(1,i); i = 2,3, \dots n\}\}$$

The vertices of degree 3, are closely connected with all remaining vertices of G. Let us consider the vertex (2,3),

$$N_{cc}(2,3) = \{(2,i); i = 1,2,4 \dots n\} \cup \{(i,3); i = 1,3,4 \dots n\}$$

and $|N_{cc}(2,3)| = 2n-1$. By using the Lemma 2.3, For any vertex $(i,j) \in V(G)$ with $N_{cc}(i,j) = 2n-1$. Then $\gamma_{cc}(G) = 1$. Let $D = \{(i,j); \deg_{cc}(i,j) = 3 \text{ and } (i,j) \in V(G)\}$. Each vertex in D is closely connected with all the remaining vertices, so $|N_{cc}(ij)| = 2n-1 \quad \forall (i,j) \in D$. Every single vertex in D is CC-dominating set. So we conclude $\gamma_{cc}(G) = 1$.

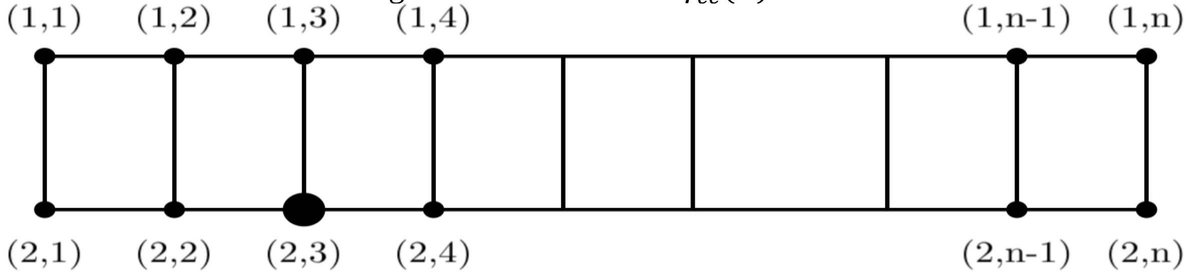

Figure 3: The vertex (2,3) form minimum CC dominating set for $P_2 \square P_n$

Theorem 2.5 *For* $n \geq 1$, $\gamma_{CC}(P_3 \square P_n) = \begin{cases} 3 & if n = 1 \\ 1 & otherwise. \end{cases}$.

Proof. Let $G = P_3 \square P_n$, we prove this result in the following two cases.

Case 1: For $n = 1$, $G = P_3 \square P_1 = P_3$, By the Theorem 2.2, we get $\gamma_{CC}(G) = 3$.

Case 2: For $n > 1$, $G = P_3 \square P_n$ and let $|V(G)| = 3n$. In G, there are four vertices of degree 2, $2n-2$ vertices are of degree 3 and $n-2$ vertices are of degree 4. Now let $S = \{(2,2), (2,3), \dots, (2, n-1)\}$. Each vertex in S have a degree 4 and $|N_{CC}(v)| = 3n-1, \forall v \in S$. By the Lemma 2.3, every vertex in S is form a CC-dominating set. Consider $D = \{(2,3)\}$ is minimum CC-dominating set. Hence $\gamma_{cc}(G) = |D| = 1$.

Example 2.6 *Consider the graph* $G = P_3 \square P_5$ *shown in Figure 4. The vertices* $S = \{(2,2), (2,3)(2,4)\}$ *each vertex in* S *have* CC *degree is 14, i.e.,* $(3 \times 5) - 1$. *Hence* $D = \{(2,3)\}$ *is a minimum CC-dominating set. So we get* $\gamma_{cc}(G) = 1$.

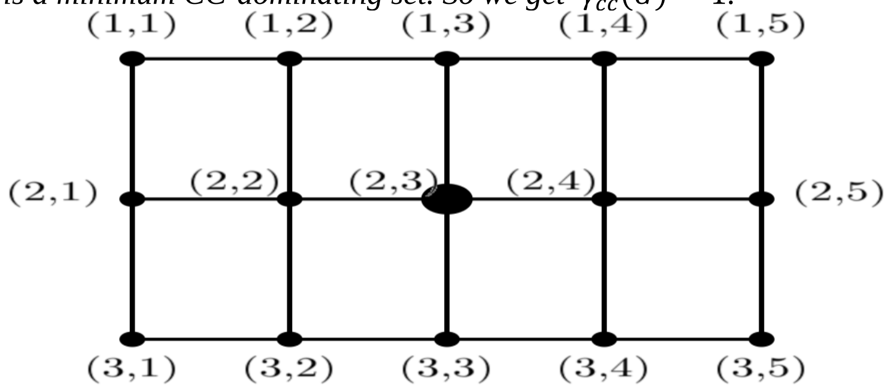

Figure 4: Vertex (2,3) form a minimum CC-dominating set for $P_3 \square P_5$

Theorem 2.7 *Let* $G = P_m \square P_n$, *then* $\gamma_{CC}(G) = \begin{cases} n & if m = 1 \\ m & if n = 1 \\ 2 & if n = 2 and m = 2 \\ 1 & Otherwise \end{cases}$

Proof. Let $G = P_m \square P_n$ and $|V(G)| = mn$. We prove this result in the following four cases.

Case 1: For $m = 1, G = P_1 \square P_n$ is a path with n vertices. Then we have $\gamma_{CC}(G) = n$.

Case 2: For $n = 1, G = P_m \square P_1$ is a also Path with m vertices. The we have $\gamma_{CC}(G) = m$.

Case 3: For $m = 2$ and $n = 2$, then $G = P_2 \square P_2$ is a cycle with four vertices. Then we get $\gamma_{CC}(C_4) = 2$.

Case 4: Let $G = P_m \square P_n$. In G the number of vertices having degree 2 is 4, $2(m+n-4)$

vertices having degree 3 and the remain vertices are having degree 4. Let $D = \{(i,j); (i,j) \in V(G)$ and $\deg(i,j) = 4\}$. Each vertex in D is closely connected with the remaining $mn - 1$ vertices. So $|N_{CC}(i,j)| = mn - 1$,for all $(i,j) \in D$ By the Lemma 2.3, Every single in D form a CC-dominating set. Hence we conclude the minimum cardinality of CC-dominating set is one. So $\gamma_{CC}(G) = 1$.

Example 2.8 *Consider the graph* $G = P_5 \square P_{10}$ *shown in Figure 5. The vertex set* $D = \{(2,2),(2,3),(2,4) \cdots (2,9)\} \cup \{(3,2),(3,3) \dots,(3,9)\} \cup \{(4,2),(4,3),\dots(4,9)\}$, *each vertex in D having degree 4. So the vertex set* $D' = \{(4,3)\}$ *is a minimum CC-dominating set. Hence* $\gamma_{CC}(P_5 \square P_{10}) = 1$.

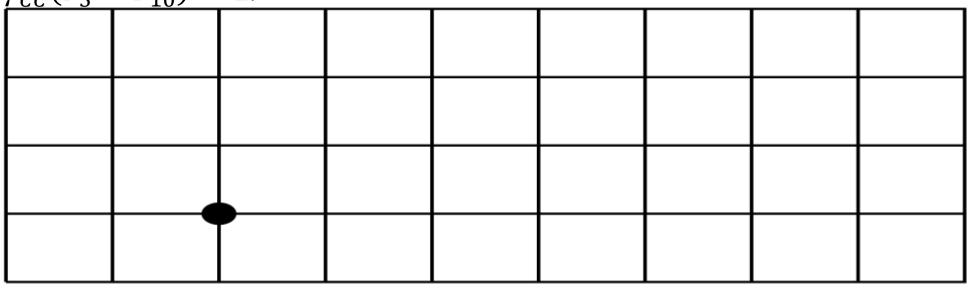

Figure 5: Vertex (4,3) form a minimum CC-dominating set for $P_5 \square P_{10}$

Corollary 2.9 *For* $m = n$, *then* $\gamma_{CC}(P_n \square P_n) = \begin{cases} 2 & if n = 2 \\ 1 & otherwise \end{cases}$

Proof. Let $G = P_n \square P_n$ and $|V(G)| = n^2$. For $n = 2$, we have $G = C_4$, then $\gamma_{CC}(P_2 \square P_2) = 2$. For $n > 2$, $G = P_n \square P_n$. By Theorem 2.7, every vertex of having degree 4 is closely connected to the remaining vertices in $P_n \square P_n$. Hence we conclude $\gamma_{CC}(P_n \square P_n) = 1$.

3 γ_{cc} in Cartesian Product of Paths and Cycle

The vertices of path and cycle can be listed in the $\{1,2,\dots,n\}$ and $\{1,2,3,\dots,n\}$ respectively. Also the edges of path are $\{(1,2)(2,3)\dots((n-1),n)\}$. The edges of cycles are $\{(1,2),(2,3),(3,4),\dots,(n-1,n),(n,1)\}$.

First we see the ordinary domination number in Cartesian product of path and cycle in the following theorem.

Theorem 3.1 *[3] For* $n \geq 3$. $\gamma(P_2 \square C_n) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil & whennisnotmultipleof 4 \\ \frac{n}{2} & whennisamultipleof 4 \end{cases}$

Theorem 3.2 *For* $n \geq 3$. $\gamma_{CC}(P_2 \square C_n) = \begin{cases} 2 & if n = 1 \\ 1 & if n > 1 \end{cases}$

Proof. Let $G = P_2 \square C_n$, $|V(G)| = 2n$. If $n = 1$, we get $G = P_2$.
By known result we conclude $\gamma_{CC}(G) = 2$.
For $n > 1$ $G = P_2 \square C_n$ and let $D = \{(i,j); (i,j) \in V(G)\}$. We get $N_{CC}(i,j) = 2n - 1, \forall (i,j) \in D$. By the Lemma 2.3 conclude $\gamma_{CC}(G) = 1$ because every vertex in D is closely connected to the remaining vertices in $V(G)$. We arrive the result.

Theorem 3.3 *For* $n \geq 3$. $\gamma_{CC}(P_3 \square C_n) = \begin{cases} 3 & if n = 1 \\ 1 & otherwise \end{cases}$

Proof. Consider the graph $G = P_3 \square C_n$ and $|V(G)| = 3n$. In the first case for $n = 1$, we arrive $G = P_3$. So we have the above result. On the other case, For $n \geq 1$, $G = P_3 \square C_n$, at least four vertices having degree 3 and at least one vertex of degree 4. Suppose $D = \{(i,j); (i,j) \in V(G)$ and $\deg(i,j) = 4\}$ and $N_{CC}(i,j) = \{V(G) - (i,j)\}$. Then $|N_{CC}(i,j)| = 3n - 1$. By using Lemma 2.3, we conclude the above result.

Theorem 3.4 *For any* $m,n > 1$, $\gamma_{CC}(P_m \square C_n) = 1$.

Proof. Let $G = P_m \square P_n$. and $|V(G)| = mn$. Let $D = \{(i,j); (i,j) \in V(G)\}$ be the vertex set of $P_m \square P_n$. Each vertex in D form a CC-dominating set, because a single vertex in D is closely connected to the remain vertices of $P_m \square P_n$. So, $|N_{CC}(i,j)| = mn - 1$. Hence we conclude that $\gamma_{CC}(G) = 1$ by using the Lemma 2.3.

Example 3.5 *The Cartesian product* $P_3 \square C_3$ *are is shown in the Figure 6. The minimum*

CC-dominating set marked as bold circle.

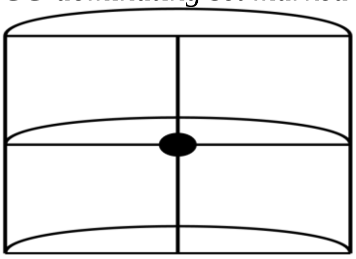

Figure 6: $P_3 \square C_3$

Theorem 3.6 *For all* $n \geq 1$, $\gamma_{CC}(C_2 \square C_n) = 1$.

Proof. Consider For $n = 1$, let $G = C_2 \square C_1$ is a cycle with two vertices then, $\gamma_{CC}(G) = 1$ and for $n > 1$, every vertex have a degree 4. Every single vertex in $C_2 \square C_n$ is closely connected to all the remain vertices, that single vertex itself form a CC-dominating set. Hence $\gamma_{CC}(C_2 \square C_n) = 1$.

Theorem 3.7 *For all* $n \geq 1$ $\gamma_{CC}(C_3 \square C_n) = 1$.

Proof. Let $G = C_3 \square C_n$. For $n = 1$, we get $G = C_3$. Then we conclude $\gamma_{CC}(G) = 1$. For $n > 1$, $G = C_3 \square C_n$. In this case every vertex is closely connected to all the remaining vertices. Then the minimum cardinality is 1. So $\gamma_{CC}(G) = 1$.

Theorem 3.8 *For* $n, m \geq 1$ $\gamma_{CC}(C_m \square C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & ifm = 1 \\ \left\lceil \frac{m}{3} \right\rceil & ifn = 1 \\ 1 & ifn > 1andm > 1 \end{cases}$

Proof. Let $G = C_m \square C_n$ and $V(G) = \{(i,j); i \leq i \leq 1, 1 \leq j \leq n\}$. We prove this result in the following cases.

Case 1: For $m = 1, G = C_n$, then By the Theorem ??, $\gamma_{CC}(G) = [\frac{n}{3}]$ and similarly for $n = 1$ we get $\gamma_{CC}(G) = [\frac{m}{3}]$.

Case 2. For $m > 1$ and $n > 1$, and $|V(G)| = mn$. In G, every vertex is at degree 4, and also we have. For each $(i,j) \in V(G)$, $|N_{CC}(i,j)| = mn - 1$. By using Lemma 2.3, we arrive the $\gamma_{CC}(G) = 1$.

Example 3.9 *Consider the Cartesian Product* $C_3 \square C_4$, *shown in the Figure 7. The vertex marked with bold dot form the* CC*-dominating set.*

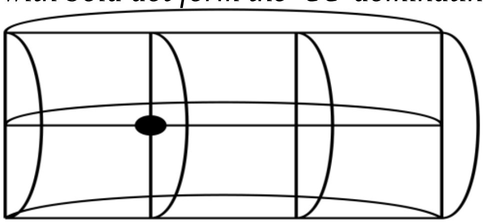

Figure 7: The Cartesian Product $C_3 \square C_4$

4 Conclusion

In this paper studied CC-domination number in Cartesian product of path, paths and cycles. Also we think that the study can be extended to some operation of two graphs Such as lexicographic and Cartesian product for general graphs is the significant direction for further research.

5 Applications

The closely connected nodes serves as a new approach to design networks and its robustness in [6]

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